The Schurman Vector
Modeling Stochastic Paths That Exhibit Mean Reversion

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The Gordon Growth Model is a valuation model that assumes that cash flows grow at a constant rate in perpetuity. With early-stage companies we often have the case where the growth rates of revenue, operating expense and capital expenditures, which are the basic components of cash flow, are not only different but are in the process of transitioning from a short-term unsustainable growth rate to a long-term sustainable growth rate. In this PDF we will develop the mathematics for The Schurman Vector. This vector employs the mathematics from Linear Algebra to build a transition matrix where expected growth rates transition from a short-term rate to a long-term rate over time. We will also develop the mathematics to add randomness to these expected paths (i.e. the expected path becomes a stochastic path). These random paths are ideal for use in Monte Carlo simulations.

Assume that we are given the short-term rate R_S and we are told that this short-term rate will decrease to the long-term rate R_L over time. This transition will be non-linear and convex to the origin. In simple terms the mathematics for this transition from the short-term rate to the long-term rate is the matrix:vector product of a 1x2 vector of current rates and a 2x2 transition matrix. We want to construct the transition matrix such that given the short-term rate R_S the long-term rate R_L can be obtained via the following linear transformation...

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{\infty} \times \begin{bmatrix} R_S \\ 0 \end{bmatrix} = \begin{bmatrix} R_L \\ R_S - R_L \end{bmatrix} \tag{1}$$

Per the linear transformation above R_S declines to R_L as time goes to infinity. With the introduction out of the way let's begin...

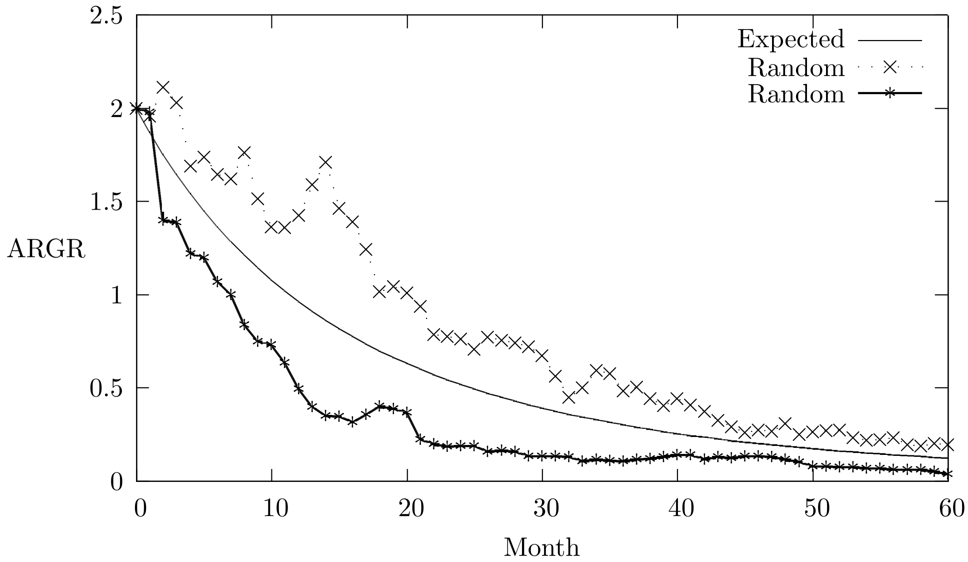
Our Hypothetical Problem

We are tasked with valuing an early-stage company with current annualized revenue of \$10 million. Current cash flow is negative due to the high level of fixed costs and capital expenditures relative to revenue. The company's short-term (actual) and long-term (projected) economics are as follows...

Description	Short-Term	Long-Term	Notes
Revenue growth rate (annualized)	200%	4%	
Ratio of expense to revenue	300%	60%	Excludes depreciation and taxes
Ratio of cap ex to revenue	150%	15%	
Ratio of net cash flow to revenue	-350%	25%	Cash flow is before tax

For this exercise we will concern ourselves with modeling revenue although the mathematics applys to the modeling of operating expense and capital expenditures. Note that when modeling the stochastic paths of revenue, expense and capital expenditures correlation will be a material factor and must be modeled.

The graph below presents three possible annualized revenue growth rate (ARGR) paths - one expected path and two random paths...



Note that expected path is smooth and the two random paths are not. What we do know is that both paths start at the short-term growth rate of 200% and eventually end up at the long-term growth rate of 4%. We know that this occurs because the transition matrix is constructed to guarantee this result. Note also that the two random paths will give us very different company values. The path that declines slowly to the long-term rate will produce significant revenue growth and the company will do very well. The path that declines rapidly to the long-term rate will most likely fail as revenue will never get to the point of cash flow break-even. To value this early-stage company we will either (1) use the expected revenue growth rate path in our valuation or (2) generate random paths and value the company via a Monte Carlo simulation.

The three paths in the chart above were constructed using the following model parameter values...

Variable	Value	Notes
x_0	2.000	Current annualized revenue growth rate
μ	0.040	Long-term annualized revenue growth rate
θ	0.400	Annual rate of mean reversion
σ	2.000	Mean reversion rate volatility (a multiplier)
Z	–	A normally-distributed random variate

An Overview Of The Mathematics

This overview is for readers who are more interested in how to use the equations rather than the mathematics used to derive them. In this section we will demonstrate the relevant equations and leave the heavy mathematics to the sections that follow.

Assume that we want to be able to model revenue by month. With that end in mind our first task will be to convert the annual rates in the parameters table above to monthly rates. The relevant annual-to-monthly rate conversions are as follows...

Variable	Value	Notes	Calculation
x_0	0.09587	Current monthly revenue growth rate	$(1+2.00)^{1/12}-1$
μ	0.00327	Long-term monthly revenue growth rate	$(1+0.04)^{1/12}-1$
θ	0.04168	Monthly rate of mean reversion	$1-(1-0.40)^{1/12}$

The table below presents the first six months of our revenue model where the variable t represents time in months, x_t is the revenue growth rate for month t, $\hat{x}_t$ is the cumulative revenue growth rate over the time interval $[0,\, t]$, $\hat{C}_t$ is annualized revenue applicable to month t and C_t is revenue earned during month t...

Mn	Month Rate	Cumul Rate	Annual Revenue	Month Revenue
t	x_t	$\hat{x}_t$	$\hat{C}_t$	C_t
1	0.09188	0.09188	10,962	914
2	0.08805	0.17993	11,971	998
3	0.08440	0.26433	13,026	1,085
4	0.08090	0.34522	14,123	1,177
5	0.07755	0.42277	15,262	1,272
6	0.07434	0.49711	16,440	1,370

The table below demonstrates the mathematics using our revenue model month six as an example (alpha is the long-term trend adjustment; Ref column refers to equation number)...

Variable	Value	Calculation	Ref
α	0.00147	$((0.00327)(0.04168)) \div (0.09587 - 0.00327)$	62
x_6	0.07434	$(0.09587)(0.00147 + 0.04168(1 - 0.00147 - 0.04168)^6)(0.00147 + 0.04168)^{-1}$	66
$\hat{x}_6$	0.49711	$(0.09587)(0.00147 + 0.04168)^{-1}((6)(0.00147) + 0.04168((1 - 0.00147 - 0.04168) -$ $(1 - 0.00147 - 0.04168)^7)(0.00147 + 0.04168)^{-1})$	70
$\hat{C}_6$	16,440	$10,000 \times Exp(0.49711)$	–
C_6	1,370	$16,440 \div 12$	–

We can add volatility to our revenue growth rate path by modeling in randomness. The table below presents the first six months of our revenue model where the added variable Z allows us to model a stochastic revenue growth rate path...

Mn	Random Number	Month X Rate	Month Y Rate	Total X and Y
t	Z	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	–	0.09587	0.00000	0.09587
1	-1.3482	0.10265	-0.00678	0.09587
2	-1.6227	0.11225	-0.01637	0.09587
3	0.1354	0.10628	-0.01040	0.09587
4	-0.7746	0.10869	-0.01282	0.09587
5	2.9344	0.07756	0.01831	0.09587
6	1.0110	0.06782	0.02805	0.09587

The table below demonstrates the mathematics using our revenue model month six as an example (the column Ref refers to equation number)...

Variable	Value	Calculation	Ref
$\bar{x}_6$	0.06782	$(1 - 0.04168(1 + (2.00)(1.0110)))(0.07756) + (0.00147)(0.01831)$	86
$\bar{y}_6$	0.02805	$0.09587 - 0.06782$	87

The demonstrate First Passage Time Equation (83) the month that the revenue growth rate will reach an annualized rate of 20% (monthly rate = 0.01531) is...

$$\begin{aligned} h &= \ln\left[\frac{(\mu + \Delta)(\alpha + \theta) - \alpha\, x_0}{\theta\, x_0}\right] \div \ln(1 - \alpha - \theta) \\ &= \ln\left[\frac{(0.01531)(0.00147 + 0.04168) - (0.00147)(0.09587)}{(0.04168)(0.09587)}\right] \div \ln(1 - 0.00147 - 0.04168) \\ &= 46.3 \end{aligned} \tag{2}$$

The Transition Matrix

We have the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{\mathbf{v}}) = \mathbf{A}\vec{\mathbf{v}}$. The matrix $\mathbf{A}$ is the transformation matrix for T with respect to the standard basis and is in the following form...

$$\mathbf{A} = \begin{bmatrix} 1-\theta & \alpha \\ \theta & 1-\alpha \end{bmatrix} \tag{3}$$

The vector $\vec{\mathbf{v}}$ is the transformed vector for T with respect to the standard basis. Vector $\vec{\mathbf{v}}$ at any time $t \geq 0$ is in the following form...

$$\vec{\mathbf{v}}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{4}$$

The vector $\vec{\mathbf{v}}$ at time $t = 1$ is the matrix:vector product of the transformation matrix as defined in Equation (3) above and the vector $\vec{\mathbf{v}}$ at time $t = 0$. In equation form vector $\vec{\mathbf{v}}$ at time $t = 1$ is...

$$\vec{\mathbf{v}}_1 = \mathbf{A}\vec{\mathbf{v}}_0 \tag{5}$$

The vector $\vec{\mathbf{v}}$ at time $t = 2$ is the matrix:vector product of our transformation matrix and the vector $\vec{\mathbf{v}}$ at time $t = 1$. In equation form vector $\vec{\mathbf{v}}$ at time $t = 2$ is...

$$\vec{\mathbf{v}}_2 = \mathbf{A}\vec{\mathbf{v}}_1 \tag{6}$$

If we substitute the right side of Equation (5) for $\vec{\mathbf{v}}_1$ in Equation (6) then we can rewrite Equation (6) as...

$$\begin{aligned} \vec{\mathbf{v}}_2 &= \mathbf{A}(\mathbf{A}\vec{\mathbf{v}}_0) \\ &= \mathbf{A}^2\,\vec{\mathbf{v}}_0 \end{aligned} \tag{7}$$

After making this substitution the vector $\vec{\mathbf{v}}$ at time $t = 2$ becomes the matrix:vector product of our transformation matrix squared and the vector $\vec{\mathbf{v}}$ at time $t = 0$. If we extend this logic to time periods greater than two it becomes apparent that at any time $t > 0$ the equation for vector $\vec{\mathbf{v}}_t$ can be written as...

$$\vec{\mathbf{v}}_t = \mathbf{A}^t\,\vec{\mathbf{v}}_0 \tag{8}$$

The vector at time zero is...

$$\vec{\mathbf{v}}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \tag{9}$$

Our goal is to transition the short-term vector $\vec{\mathbf{v}}_0$ as defined by Equation (9) above to the long-term vector $\vec{\mathbf{v}}_\infty$ such that...

$$\begin{gathered} \lim_{t \to \infty} \mathbf{A}^t\,\vec{\mathbf{v}}_0 = \vec{\mathbf{v}}_t \\ \begin{bmatrix} 1-\theta & \alpha \\ \theta & 1-\alpha \end{bmatrix}^\infty \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu \\ x_0 - \mu \end{bmatrix} \end{gathered} \tag{10}$$

As noted above vector $\vec{\mathbf{v}}$ at time t can be written as the matrix:vector product of our transformation matrix and vector $\vec{\mathbf{v}}$ at time $t-1$. Equation (8) can therefore be rewritten as...

$$\begin{aligned} \vec{\mathbf{v}}_t &= \mathbf{A}\vec{\mathbf{v}}_{t-1} \\ \begin{bmatrix} x_t \\ y_t \end{bmatrix} &= \begin{bmatrix} 1-\theta & \alpha \\ \theta & 1-\alpha \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} \\ \begin{bmatrix} x_t \\ y_t \end{bmatrix} &= \begin{bmatrix} (1-\theta)\,x_{t-1} + \alpha\,y_{t-1} \\ \theta\,x_{t-1} + (1-\alpha)\,y_{t-1} \end{bmatrix} \end{aligned} \tag{11}$$

Per Equation (11) we have the following two linear equations...

$$x_t = (1-\theta)\,x_{t-1} + \alpha\,y_{t-1} \tag{12}$$

$$y_t = \theta\,x_{t-1} + (1-\alpha)\,y_{t-1} \tag{13}$$

The sum of Equations (12) and (13) is...

$$\begin{aligned} x_t + y_t &= (1-\theta)\, x_{t-1} + \alpha\, y_{t-1} + \theta\, x_{t-1} + (1-\alpha)\, y_{t-1} \\ &= x_{t-1} + y_{t-1} \end{aligned} \tag{14}$$

Note that per Equation (14) the sum of the elements in vector $\vec{\mathbf{v}}_t$ does not change from one period to another. This means that the sum of the elements in $\vec{\mathbf{v}}_t$ is always equal to the sum of the elements in $\vec{\mathbf{v}}_0$ as defined by Equation (9) above. In other words the following equality always holds...

$$x_t + y_t = x_0 \tag{15}$$

Remember that per Equation (10) above our goal is to transition vector $\vec{\mathbf{v}}_0$ to vector $\vec{\mathbf{v}}_\infty$ over time. As time $t \to \infty$ we want vector $\vec{\mathbf{v}}_0$ to become...

$$\mathbf{A}^\infty\, \vec{\mathbf{v}}_0 = \vec{\mathbf{v}}_\infty = \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} \mu \\ x_0 - \mu \end{bmatrix} \tag{16}$$

There are two variables in our transition matrix as defined by Equation (3) above. The first variable is θ, which is the periodic rate of mean reversion. This variable is an exogenous variable given to us subject to constraints. The second variable is α, which is the periodic long-term adjustment factor. This variable is an endogenous variable and therefore must be calculated. How do we set the value of α such that the transition described in Equation (16) takes place?

To determine α we will proceed as follows:

Step 1	-	Determine the eigenbasis for our transition matrix as defined by Equation (3).
Step 2	-	Perform a change of basis from the standard basis to the eigenbasis determined in Step 1.
Step 3	-	Determine the value of α such that the transition described in Equation (16) is accomplished.

The Eigenbasis as a New Coordinate System

Eigenvalues and eigenvectors are critical to the understanding of the long-run properties of Markov chains. The eigenbasis for our transformation matrix as defined in Equation (3) will consist of two eigenvalues, which we will define as λ_1 and λ_2, and two eigenvectors, which we will define as $\vec{\mathbf{z}}_1$ and $\vec{\mathbf{z}}_2$. For the eigenvectors to span $\mathbb{R}^2$ the eigenvectors $\vec{\mathbf{z}}_1$ and $\vec{\mathbf{z}}_2$ must be linearly independent.

If matrix $\mathbf{A}$ is a square matrix, which in our case it is, a non-zero vector $\vec{\mathbf{z}}$ is an eigenvector of $\mathbf{A}$ if there is a scalar λ such that...

$$\mathbf{A}\vec{\mathbf{z}} = \lambda\vec{\mathbf{z}} \tag{17}$$

The linear transformation T described above has matrix $\mathbf{A}$ and vector $\vec{\mathbf{v}}$ as linear transformations with respect to the standard basis. The standard basis, which we will define as S, in $\mathbb{R}^2$ is...

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{18}$$

Our goal here is to find the two eigenvectors of $\mathbf{A}$ so that we can create a new basis B for the linear transformation T. We will then perform a change of basis from the standard basis S to the eigenbasis B. Once this change of basis has been made we can set the value of α, which is the periodic long-run adjustment factor, such that Equation (16) holds.

Our first step will be to rearrange Equation (17) such that the zero vector $\vec{\mathbf{0}}$ is on the right hand side of the equality. Noting that matrix $\mathbf{I}_2$ is the identity matrix in $\mathbb{R}^2$, Equation (17) becomes...

$$\begin{aligned} \mathbf{A}\vec{\mathbf{z}} &= \lambda\vec{\mathbf{z}} \\ \mathbf{A}\vec{\mathbf{z}} &= \lambda\mathbf{I}_2\vec{\mathbf{z}} \\ \lambda\mathbf{I}_2\vec{\mathbf{z}} - \mathbf{A}\vec{\mathbf{z}} &= \vec{\mathbf{0}} \\ (\lambda\mathbf{I}_2 - \mathbf{A})\vec{\mathbf{z}} &= \vec{\mathbf{0}} \end{aligned} \tag{19}$$

The eigenvectors that we seek are non-zero vectors by definition. Note that vector $\vec{\mathbf{z}}$ is a non-zero vector only if the square matrix $\lambda\mathbf{I}_2 - \mathbf{A}$ cannot be inverted, which means that the determinant of this matrix must be zero. The matrix on the left hand side of Equation (19) can be rewritten as...

$$\begin{aligned}\lambda\mathbf{I_2} - \mathbf{A} &= \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1-\theta & \alpha\\ \theta & 1-\alpha\end{bmatrix}\\ &= \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}1-\theta & \alpha\\ \theta & 1-\alpha\end{bmatrix}\\ &= \begin{bmatrix}\lambda+\theta-1 & -\alpha\\ -\theta & \lambda+\alpha-1\end{bmatrix}\end{aligned} \tag{20}$$

Noting that the determinant of Equation (20) must be zero, the characteristic polynomial $\Delta(\lambda)$ of the transformation matrix $\mathbf{A}$ is...

$$\begin{aligned}\Delta(\lambda) &= |\lambda\mathbf{I_2} - \mathbf{A}|\\ &= [(\lambda+\theta-1)(\lambda+\alpha-1)] - [(-\alpha)(-\theta)]\\ &= \lambda^2 + \lambda\alpha - \lambda + \lambda\theta + \alpha\theta - \theta - \lambda - \alpha + 1 - \alpha\theta\\ &= \lambda^2 + \lambda(\alpha+\theta-2) + (1-\alpha-\theta)\end{aligned} \tag{21}$$

If we make the following definitions...

$$a = 1 \ \text{...}\textit{and}\text{...} \ b = \alpha+\theta-2 \ \text{...}\textit{and}\text{...} \ c = 1-\alpha-\theta \tag{22}$$

Then Equation (21) becomes...

$$\Delta(\lambda) = a\lambda^2 + b\lambda + c \tag{23}$$

The eigenvalues of matrix $\mathbf{A}$ are found by setting Equation (23) to zero and solving for the two roots using the definitions provided in Equation (22). The two eigenvalues of matrix $\mathbf{A}$ are...

$$\begin{aligned}\lambda &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\ &= \frac{-(\alpha+\theta-2) \pm \sqrt{(\alpha+\theta-2)^2 - (4)(1)(1-\alpha-\theta)}}{(2)(1)}\\ &= \frac{2-\alpha-\theta \pm \sqrt{\alpha^2+\theta^2+2\alpha\theta}}{2}\\ &= \frac{2-\alpha-\theta \pm \sqrt{(\alpha+\theta)^2}}{2}\\ &= \frac{2-\alpha-\theta \pm (\alpha+\theta)}{2}\end{aligned} \tag{24}$$

Using Equation (24) the eigenvalue λ_1 is...

$$\lambda_1 = \frac{2-\alpha-\theta+(\alpha+\theta)}{2} = 1 \tag{25}$$

Using Equation (24) the eigenvalue λ_2 is...

$$\lambda_2 = \frac{2-\alpha-\theta-(\alpha+\theta)}{2} = 1-\alpha-\theta \tag{26}$$

To find the eigenvectors of matrix $\mathbf{A}$ given the eigenvalues determined above we need to solve Equation (19) for each eigenvalue. Equation (19) rewritten as a system of linear equations is...

$$\begin{gathered}(\lambda\mathbf{I_2} - \mathbf{A})\vec{\mathbf{z}} = \mathbf{0}\\ \begin{bmatrix}\lambda+\theta-1 & -\alpha\\ -\theta & \lambda+\alpha-1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}\end{gathered} \tag{27}$$

Given the first eigenvalue $\lambda_1 = 1$ as defined by Equation (25) the system of linear equations that must be solved so as to find the first eigenvector $\vec{\mathbf{z}}_1$ is...

$$\begin{aligned} \begin{bmatrix} [1] + \theta - 1 & -\alpha \\ -\theta & [1] + \alpha - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \theta & -\alpha \\ -\theta & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{28}$$

The two simultaneous equations from Equation (28) are...

$$\theta x - \alpha y = 0 \ \ \textit{...and...} \ \ -\theta x + \alpha y = 0 \tag{29}$$

The solution to the system of linear equations in Equation (29) is...

$$\begin{aligned} \theta x - \alpha y &= -\theta x + \alpha y \\ 2\,\theta x &= 2\,\alpha y \\ x &= \frac{\alpha}{\theta}\,y \end{aligned} \tag{30}$$

Using the definition supplied by Equation (30) the first eigenvector $\vec{\mathbf{z}}_1$ associated with the first eigenvalue λ_1 is...

$$\vec{\mathbf{z}}_1 = \begin{bmatrix} \frac{\alpha}{\theta} \\ 1 \end{bmatrix} \tag{31}$$

The eigenspace E_{λ_1} associated with the first eigenvalue as defined by Equation (25) and the eigenvector as defined by Equation (31) is...

$$E_{\lambda=1} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} \frac{\alpha}{\theta} \\ 1 \end{bmatrix} \middle| \, t \,\epsilon\, \mathbb{R} \right\} \tag{32}$$

Given the second eigenvalue $\lambda_2 = 1 - \alpha - \theta$ as defined by Equation (26) the system of linear equations that must be solved so as to find the second eigenvector $\vec{\mathbf{z}}_2$ is...

$$\begin{aligned} \begin{bmatrix} (1 - \alpha - \theta) + \theta - 1 & -\alpha \\ -\theta & (1 - \alpha - \theta) + \alpha - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\alpha & -\alpha \\ -\theta & -\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{33}$$

The two simultaneous equations from Equation (33) are...

$$-\alpha x - \alpha y = 0 \ \ \textit{...and...} \ \ -\theta x - \theta y = 0 \tag{34}$$

The solution to the system of linear equations in Equation (34) is...

$$\begin{aligned} -\alpha x - \alpha y &= -\theta x - \theta y \\ -(\alpha + \theta)x &= (\alpha + \theta)y \\ x &= -y \end{aligned} \tag{35}$$

Using the definition supplied by Equation (35) the second eigenvector $\vec{\mathbf{z}}_2$ associated with the second eigenvalue λ_2 is...

$$\vec{\mathbf{z}}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{36}$$

The eigenspace E_{λ_2} associated with the second eigenvalue as defined by Equation (26) and the eigenvector as defined by Equation (36) is...

$$E_{\lambda=1-\alpha-\theta} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \middle| \, t \,\epsilon\, \mathbb{R} \right\} \tag{37}$$

The new basis for the linear transformation consists of the eigenvectors in Equations (32) and (37) above provided that these column vectors are linearly independent (i.e the eigenvectors span $\mathbb{R}^2$). The new basis, which is also known as the eigenbasis, is...

$$B = \left\{ \begin{bmatrix} \frac{\alpha}{\theta} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \tag{38}$$

As noted above the column vectors for our new basis as described by Equation (38) is a valid basis if and only if the column vectors are linearly independent. The vectors are independent if the solution to c_1 and c_2 in the equation below is non-zero...

$$\begin{bmatrix} \frac{\alpha}{\theta} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{39}$$

If we multiply the vector of unknowns by the eigenbasis matrix in Equation (39) the two linear equations that need to be solved are...

$$c_1 \frac{\alpha}{\theta} - c_2 = 0 \tag{40}$$

$$c_1 + c_2 = 0 \tag{41}$$

If we add Equations (40) and (41) so as to eliminate c_2 we get...

$$c_1 (1 + \frac{\alpha}{\theta}) = 0 \tag{42}$$

Per Equation (42) as long as the ratio of α to θ is not negative one then $c_1 = 0$. Per Equation (41) if $c_1 = 0$ then $c_2 = 0$. If both c_1 and c_2 are zero then the columns in basis B are linearly independent and therefore B is a valid basis. For B to be a valid basis then the following condition must be met...

$$\frac{\alpha}{\theta} \neq -1 \tag{43}$$

Change Basis to the Eigenbasis and Solve for Alpha

The change of basis requires that we convert our beginning vector $\vec{\mathbf{v}}_0$ as defined by Equation (9), our ending vector $\vec{\mathbf{v}}_\infty$ as defined by Equation (16), and our transformation matrix $\mathbf{A}$ as defined by Equation (3) from the standard basis S to the eigenbasis B. We will accomplish this via the utilization of a change of basis matrix which is a square matrix that consists of the column vectors of our new basis as defined in Equation (38) above. The change of basis matrix $\mathbf{C}$ is therefore...

$$\mathbf{C} = \begin{bmatrix} \frac{\alpha}{\theta} & -1 \\ 1 & 1 \end{bmatrix} \tag{44}$$

To perform the change of basis we need the determinant and inverse of the change of basis matrix $\mathbf{C}$ as defined by Equation (44) above. The determinant of matrix $\mathbf{C}$ is...

$$|C| = \left(\frac{\alpha}{\theta}\right)(1) - (1)(-1) = \frac{\alpha}{\theta} + 1 = \frac{\alpha + \theta}{\theta} \tag{45}$$

The inverse of matrix $\mathbf{C}$ using the determinant as defined by Equation (45) above is...

$$\mathbf{C}^{-1} = \frac{1}{|C|} \begin{bmatrix} 1 & 1 \\ -1 & \frac{\alpha}{\theta} \end{bmatrix} = \frac{\theta}{\alpha + \theta} \begin{bmatrix} 1 & 1 \\ -1 & \frac{\alpha}{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\theta}{\alpha+\theta} & \frac{\theta}{\alpha+\theta} \\ \frac{-\theta}{\alpha+\theta} & \frac{\alpha}{\alpha+\theta} \end{bmatrix} \tag{46}$$

Our beginning vector at time zero as defined by Equation (9) converted to the new basis B is...

$$[\vec{\mathbf{v}}_0]_B = \mathbf{C}^{-1} \, \vec{\mathbf{v}}_0 = \begin{bmatrix} \frac{\theta}{\alpha+\theta} & \frac{\theta}{\alpha+\theta} \\ \frac{-\theta}{\alpha+\theta} & \frac{\alpha}{\alpha+\theta} \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta} x_0 \end{bmatrix} \tag{47}$$

Our ending vector at time infinity as defined by Equation (16) converted to the new basis B is...

$$[\vec{\mathbf{v}}_\infty]_B = \mathbf{C}^{-1} \, \vec{\mathbf{v}}_\infty = \begin{bmatrix} \frac{\theta}{\alpha+\theta} & \frac{\theta}{\alpha+\theta} \\ \frac{-\theta}{\alpha+\theta} & \frac{\alpha}{\alpha+\theta} \end{bmatrix} \begin{bmatrix} \mu \\ x_0 - \mu \end{bmatrix} = \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta} \mu + \frac{\alpha}{\alpha+\theta} (x_0 - \mu) \end{bmatrix} \tag{48}$$

We will define matrix $\mathbf{D}$ as the transformation matrix $\mathbf{A}$ converted to the new basis B. The standard way of converting a matrix from one basis to another is via the following equation...

$$\mathbf{D} = \mathbf{C}^{-1} \mathbf{A} \, \mathbf{C} \tag{49}$$

Our transformation matrix $\mathbf{A}$ multiplied by the change of basis matrix $\mathbf{C}$ can be written as matrix $\mathbf{A}$ times the basis vectors in matrix $\mathbf{C}$ as defined by the following equation...

$$\mathbf{A}\,\mathbf{C} = \begin{bmatrix} \mathbf{A}\vec{\mathbf{z}}_1 & \mathbf{A}\vec{\mathbf{z}}_2 \end{bmatrix} \tag{50}$$

Note that per the definition of eigenvalues and eigenvectors the following definitions hold...

$$\mathbf{A}\vec{\mathbf{z}}_1 = \lambda_1 \vec{\mathbf{z}}_1 \tag{51}$$

$$\mathbf{A}\vec{\mathbf{z}}_2 = \lambda_2 \vec{\mathbf{z}}_2 \tag{52}$$

After substituting Equations (51) and (52) into Equation (50) we get...

$$\mathbf{A}\,\mathbf{C} = \begin{bmatrix} \lambda_1\vec{\mathbf{z}}_1 & \lambda_2\vec{\mathbf{z}}_2 \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{z}}_1 & \vec{\mathbf{z}}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \mathbf{C} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{53}$$

After substituting Equation (53) into Equation (49) we get the definition of the matrix $\mathbf{D}$ which is...

$$\mathbf{D} = \mathbf{C}^{-1}\mathbf{A}\,\mathbf{C} = \mathbf{C}^{-1}\mathbf{C} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{54}$$

Drop in the values of lambda...

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\theta \end{bmatrix} \tag{55}$$

Matrix $\mathbf{D}$ raised to the t power is...

$$\mathbf{D}^t = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\theta \end{bmatrix}^t = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\theta)^t \end{bmatrix} \tag{56}$$

Per the matrix above we have the numbers 1 and $1-\alpha-\theta$ across the diagonals and zeros elsewhere. We will make the following constraint definition...

$$0 < 1-\alpha-\theta < 1 \tag{57}$$

Note that per the constraint in Equation (57) above $1-\alpha-\theta$ is greater than zero and less than one. When matrix $\mathbf{D}$ is raised to an infinite power matrix element a_{11} (λ_1) stays at one while matrix element a_{22} (λ_2) goes to zero such that matrix $\mathbf{D}$ becomes...

$$\mathbf{D}^\infty = \lim_{t\to\infty} \mathbf{D}^t = \lim_{t\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\theta \end{bmatrix}^t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{58}$$

Our vector at time infinity as defined by Equation (48) as a function of the transformation matrix $\mathbf{D}$ at time infinity as defined in Equation (58) and our vector at time zero as defined by Equation (47) is...

$$\begin{aligned} \mathbf{D}^\infty[\vec{\mathbf{v}}_0]_B &= [\vec{\mathbf{v}}_\infty]_B \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta}\, x_0 \end{bmatrix} &= \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta}\,\mu + \frac{\alpha}{\alpha+\theta}\,(x_0-\mu) \end{bmatrix} \\ \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta}\,\mu + \frac{\alpha}{\alpha+\theta}\,(x_0-\mu) \end{bmatrix} \end{aligned} \tag{59}$$

We can now solve for alpha. Equation (59) gives us the following simulaneous equations...

$$\frac{\theta}{\alpha+\theta} x_0 = \frac{\theta}{\alpha+\theta} x_0 \tag{60}$$

$$0 = \frac{-\theta}{\alpha+\theta}\,\mu + \frac{\alpha}{\alpha+\theta}\,(x_0-\mu) \tag{61}$$

Note that Equation (60) above does not need to be solved because $x_0 + y_0$ are on both sides of the equation. The solution to Equation (61) in terms of the undefined variable alpha is...

$$\begin{aligned} \frac{-\theta}{\alpha+\theta}\,\mu + \frac{\alpha}{\alpha+\theta}\,(x_0-\mu) &= 0 \\ -\theta\,\mu + \alpha\,(x_0-\mu) &= 0 \\ \alpha\,(x_0-\mu) &= \mu\,\theta \\ \alpha &= \frac{\mu\,\theta}{x_0-\mu} \end{aligned} \tag{62}$$

The Vector Values of X and Y

Per Equation (8) above vector $\vec{\mathbf{v}}$ at time t can be determined by taking the matrix-vector product of matrix $\mathbf{A}$ to the t'th power and vector $\vec{\mathbf{v}}$ at time zero. This linear transformation is done in the standard basis S. The first and second elements of the transformed vector $\vec{\mathbf{v}}_t$ are x_t and y_t, respectively. We can also determine vector $\vec{\mathbf{v}}$ at time t via the following steps...

Step 1	-	Convert $\vec{\mathbf{v}}_0$ from the standard basis S to the eigenbasis B per Equation (47)
Step 2	-	Perform the linear transformation using matrix $\mathbf{D}$ as defined by Equation (54)
Step 3	-	Convert the transformed vector back to the standard basis S

This $S \Rightarrow B \Rightarrow S$ operation in equation form is...

$$\vec{\mathbf{v}}_t = \mathbf{C}\Big[\mathbf{D}^t\Big[\mathbf{C}^{-1}\,\vec{\mathbf{v}}_0\Big]\Big] \tag{63}$$

The matrix vector product of matrix $\mathbf{D}$ raised to the t power, as defined by Equation (56), and vector $\vec{\mathbf{v}}$ at time zero converted to the eigenbasis, as defined by Equation (47), is...

$$\mathbf{D}^t\Big[\mathbf{C}^{-1}\,\vec{\mathbf{v}}_0\Big] = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\theta)^t \end{bmatrix} \begin{bmatrix} \frac{\theta}{\alpha+\theta}x_0 \\ \frac{-\theta}{\alpha+\theta}\,x_0 \end{bmatrix} = \begin{bmatrix} x_0\,\theta\,(\alpha+\theta)^{-1} \\ -x_0\,\theta\,(1-\alpha-\theta)^t\,(\alpha+\theta)^{-1} \end{bmatrix} \tag{64}$$

The matrix vector product of $\mathbf{C}$, as defined by Equation (44), and the vector $\mathbf{D}^t[\mathbf{C}^{-1}\,\vec{\mathbf{v}}_0]$ as defined by Equation (64) above, is...

$$\begin{aligned} \mathbf{C}\Big[\mathbf{D}^t\Big[\mathbf{C}^{-1}\,\vec{\mathbf{v}}_0\Big]\Big] &= \begin{bmatrix} \frac{\alpha}{\theta} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0\,\theta\,(\alpha+\theta)^{-1} \\ -x_0\,\theta\,(1-\alpha-\theta)^t\,(\alpha+\theta)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} x_0\,\alpha\,(\alpha+\theta)^{-1} + x_0\,\theta\,(1-\alpha-\theta)^t(\alpha+\theta)^{-1} \\ x_0\,\theta\,(\alpha+\theta)^{-1} - x_0\,\theta\,(1-\alpha-\theta)^t\,(\alpha+\theta)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} x_0\,(\alpha+\theta\,(1-\alpha-\theta)^t)(\alpha+\theta)^{-1} \\ x_0\,(\theta-\theta\,(1-\alpha-\theta)^t)(\alpha+\theta)^{-1} \end{bmatrix} \end{aligned} \tag{65}$$

From Equation (65) above we have two equations that define the values of x_t and y_t, respectively. The equation for x_t, which is vector $\vec{\mathbf{v}}$ element one in the standard basis S, is...

$$x_t = x_0\,(\alpha+\theta\,(1-\alpha-\theta)^t)(\alpha+\theta)^{-1} \tag{66}$$

The equation for y_t, which is vector $\vec{\mathbf{v}}$ element two in the standard basis S, is...

$$y_t = x_0\,(\theta-\theta\,(1-\alpha-\theta)^t)(\alpha+\theta)^{-1} \tag{67}$$

Per Equation (15) above the sum of x_t and y_t will always equal x_0. Rather than use Equation (67) as the definition of y_t we will use the following definition of y_t for the sake of computational convenience...

$$y_t = x_0 - x_t \tag{68}$$

The Cumulative Values of X and Y

We want an equation for the cumulative value of x over the time interval $[1, \mathrm{t}]$ where $t \geq 1$. Using the definition of x in Equation (66) above the equation for the cumulative value of x over the aforementioned time interval is...

$$\begin{aligned}
\hat{x}_t &= \sum_{i=1}^{t} x_i \\
&= \sum_{i=1}^{t} x_0\,(\alpha + \theta\,(1-\alpha-\theta)^i)(\alpha+\theta)^{-1} \\
&= \sum_{i=1}^{t} x_0\,\alpha\,(\alpha+\theta)^{-1} + \sum_{i=1}^{t} x_0\,\theta\,(1-\alpha-\theta)^i(\alpha+\theta)^{-1} \\
&= x_0\,\alpha\,(\alpha+\theta)^{-1}\sum_{i=1}^{t} i + x_0\,\theta\,(\alpha+\theta)^{-1}\sum_{i=1}^{t}(1-\alpha-\theta)^i \\
&= t\,x_0\,\alpha\,(\alpha+\theta)^{-1} + x_0\,\theta\,(\alpha+\theta)^{-1}\sum_{i=1}^{t}(1-\alpha-\theta)^i
\end{aligned} \tag{69}$$

After replacing the summation in Equation (69) with Equation (93) the equation for the cumulative value of x over the time interval $[1, \mathrm{t}]$ becomes...

$$\begin{aligned}
\hat{x}_t &= t\,x_0\,\alpha\,(\alpha+\theta)^{-1} + x_0\,\theta\,(\alpha+\theta)^{-1}((1-\alpha-\theta) - (1-\alpha-\theta)^{t+1})\,(\alpha+\theta)^{-1} \\
&= x_0\,(\alpha+\theta)^{-1}\,(t\,\alpha + \theta\,((1-\alpha-\theta) - (1-\alpha-\theta)^{t+1})\,(\alpha+\theta)^{-1})
\end{aligned} \tag{70}$$

Using the definition of y in Equation (68) above the equation for the cumulative value of y over the time interval $[1, \mathrm{t}]$ is...

$$\begin{aligned}
\hat{y}_t &= \sum_{i=1}^{t}\Big[x_0 - x_i\Big] \\
&= \sum_{i=1}^{t} x_0 - \sum_{i=1}^{t} x_i \\
&= t\,x_0 - \hat{x}_t
\end{aligned} \tag{71}$$

The Derivatives of X and Y

The first and second derivatives of vector values x_t and y_t will define the shape of our curves. The first derivative of vector $\vec{\mathbf{v}}$ element one as defined by Equation (66) is...

$$\frac{\delta x_t}{\delta t} = x_0\,\theta\ln(1-\alpha-\theta)(1-\alpha-\theta)^t(\alpha+\theta)^{-1} \tag{72}$$

The second derivative of vector $\vec{\mathbf{v}}$ element one as defined by Equation (66) is...

$$\frac{\delta^2 x_t}{\delta t^2} = x_0\,\theta\ln(1-\alpha-\theta)^2(1-\alpha-\theta)^t(\alpha+\theta)^{-1} \tag{73}$$

The first derivative of vector $\vec{\mathbf{v}}$ element two as defined by Equation (68) is...

$$\frac{\delta y_t}{\delta t} = -\frac{\delta x_t}{\delta t} \tag{74}$$

The second derivative of vector $\vec{\mathbf{v}}$ element two as defined by Equation (68) is...

$$\frac{\delta^2 y_t}{\delta t^2} = -\frac{\delta^2 x_t}{\delta t^2} \tag{75}$$

Note that if $0 < 1 - \alpha - \theta < 1$ then the first derivative of x is negative and the second derivative of x is positive, which means that x decreases over time but at a decreasing rate.

First Passage Time

Remember that vector $\vec{\mathbf{v}}$ at times zero (Equation (9)) and infinity (Equation (16)) are...

$$\vec{\mathbf{v}}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \quad ...and... \quad \vec{\mathbf{v}}_\infty = \begin{bmatrix} \mu \\ x_0 - \mu \end{bmatrix} \tag{76}$$

We want to calculate the amount of time that it would take x_0 to hit some target rate $\mu + \Delta$. We want to calculate the value of vector subscript h such that vector $\vec{\mathbf{v}}_0$ in Equation (9) ends up in the following form by time h...

$$\vec{\mathbf{v}}_h = \begin{bmatrix} \mu + \Delta \\ x_0 - \mu - \Delta \end{bmatrix} \tag{77}$$

The value of subscript h is the **first passage time**. We will use first passage time to calibrate our curves. To make this calculation we will again make use of the Eigenbasis B as described above. The first step in the calculation will be to convert our beginning vector $\vec{\mathbf{v}}_0$, which is in the standard basis S, to the eigenbasis B as was done in Equation (47) above. Vector $\vec{\mathbf{v}}_0$ converted to the eigenbasis is...

$$[\vec{\mathbf{v}}_0]_B = \mathbf{C}^{-1}\mathbf{v}_0 = \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta}\, x_0 \end{bmatrix} \tag{78}$$

The second step in the calculation will be to convert our ending vector $\vec{\mathbf{v}}_h$, which is in the standard basis S and is defined by Equation (77) above, to the eigenbasis B. Using the inverse of our change of basis matrix $\mathbf{C}$ as defined by Equation (46), vector $\vec{\mathbf{v}}_h$ converted to the eigenbasis is...

$$[\vec{\mathbf{v}}_h]_B = \mathbf{C}^{-1}\mathbf{v}_0 = \begin{bmatrix} \frac{\theta}{\alpha+\theta} & \frac{\theta}{\alpha+\theta} \\ \frac{-\theta}{\alpha+\theta} & \frac{\alpha}{\alpha+\theta} \end{bmatrix} \begin{bmatrix} \mu + \Delta \\ x_0 - \mu - \Delta \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\alpha+\theta}\, x_0 \\ \frac{\alpha}{\alpha+\theta}\, x_0 - (\mu + \Delta) \end{bmatrix} \tag{79}$$

The third step is to set up the linear transformation using matrix $\mathbf{D}$ as defined by Equation (56). The linear transformation in equation form is...

$$\begin{gathered} \mathbf{D}^h [\vec{\mathbf{v}}_0]_B = [\vec{\mathbf{v}}_h]_B \\ \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\theta)^h \end{bmatrix} \begin{bmatrix} \frac{\theta}{\alpha+\theta} x_0 \\ \frac{-\theta}{\alpha+\theta}\, x_0 \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\alpha+\theta}\, x_0 \\ \frac{-\theta}{\alpha+\theta}\, (1-\alpha-\theta)^h x_0 \end{bmatrix} \end{gathered} \tag{80}$$

The system of linear equations represented by Equation (80) above is...

$$\frac{\theta}{\alpha+\theta}\, x_0 = \frac{\theta}{\alpha+\theta}\, x_0 \tag{81}$$

$$\frac{-\theta}{\alpha+\theta}\, (1-\alpha-\theta)^h x_0 = \frac{\alpha}{\alpha+\theta}\, x_0 - (\mu + \Delta) \tag{82}$$

The first equation (Equation (81)) does not need to be solved because the right and left sides of the equality are the same. We can determine first passage time by solving the second equation (Equation (82)) in terms of h. The equation for first passage time is therefore...

$$\begin{aligned} \frac{-\theta}{\alpha+\theta}\, (1-\alpha-\theta)^h x_0 &= \frac{\alpha}{\alpha+\theta}\, x_0 - (\mu + \Delta) \\ -\theta(1-\alpha-\theta)^h x_0 &= \alpha\, x_0 - (\mu + \Delta)(\alpha + \theta) \\ (1-\alpha-\theta)^h &= \frac{(\mu + \Delta)(\alpha + \theta) - \alpha\, x_0}{\theta\, x_0} \\ h\, \ln(1-\alpha-\theta) &= \ln\left[\frac{(\mu + \Delta)(\alpha + \theta) - \alpha\, x_0}{\theta\, x_0}\right] \\ h\, &= \ln\left[\frac{(\mu + \Delta)(\alpha + \theta) - \alpha\, x_0}{\theta\, x_0}\right] \div \ln(1-\alpha-\theta) \end{aligned} \tag{83}$$

Stochastic Paths With Mean Reversion

As described in the sections above the vector $\vec{\mathbf{v}}_{t+1}$ can be written as the matrix-vector product of our transformation matrix $\mathbf{A}$ and $\vec{\mathbf{v}}_t$. From the vantage point of time t vector $\vec{\mathbf{v}}_t$ is known at time t and vector $\vec{\mathbf{v}}_{t+1}$ is unknown at time t. To model the expected growth rate path as a stochastic path we will redefine the transformation matrix $\mathbf{A}$ as defined by Equation (3) above as..

$$\bar{\mathbf{A}} = \begin{bmatrix} 1-\theta\,(1+\sigma\,Z) & \alpha \\ \theta\,(1+\sigma\,Z) & 1-\alpha \end{bmatrix} \;...where...\;\; Z \sim N[0,1] \tag{84}$$

From the vantage point of time t vector $\vec{\mathbf{v}}_{t+1}$ is a linear transformation of vector $\vec{\mathbf{v}}_t$ and the redefined transition matrix $\bar{\mathbf{A}}$. In equation form this transformation is...

$$\begin{aligned} \vec{\mathbf{v}}_{t+1} &= \bar{\mathbf{A}}\vec{\mathbf{v}}_1 \\ \begin{bmatrix} \bar{x}_{t+1} \\ \bar{y}_{t+1} \end{bmatrix} &= \begin{bmatrix} 1-\theta\,(1+\sigma\,Z) & \alpha \\ \theta\,(1+\sigma\,Z) & 1-\alpha \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \end{aligned} \tag{85}$$

Note that because at time t we don't know the values for x_{t+1} and y_{t+1} these vector elements are written as $\bar{x}_{t+1}$ and $\bar{y}_{t+1}$, respectively.

The speed at which the short-term unsustainable growth rate declines to the long-term sustainable growth rate is a function of the rate of mean reversion (θ). If this mean reversion rate is greater than expected then the short-term rate will deline to the long-term rate faster than expected. If the mean reversion rate is less than expected the short-term rate will deline to the long-term rate slower than expected. By adding the $(1+\sigma\,Z)$ multiplier in the matrix above we have converted the deterministic mean reversion rate to a stochastic mean reversion rate. Per Equation (85) above we have the following two stochastic linear equations...

$$\bar{x}_{t+1} = (1-\theta\,(1+\sigma\,Z))\,x_t + \alpha\,y_t \tag{86}$$

$$\bar{y}_{t+1} = x_0 - \bar{x}_{t+1} \tag{87}$$

The mean and variance of the random growth rate $\hat{x}_{t+1}$ per Appendix Equations (94) and (95) are...

$$mean = \mathbb{E}\Big[\bar{x}_{t+1}\Big] = (1-\theta)\,x_t + \alpha\,y_t \tag{88}$$

$$variance = \mathbb{E}\Big[\bar{x}^2_{t+1}\Big] - \Big\{\mathbb{E}\Big[\bar{x}_{t+1}\Big]\Big\}^2 = \theta^2\sigma^2x_t^2 \tag{89}$$

The mean and variance of the random growth rate $\bar{y}_{t+1}$ per Appendix Equations (96) and (97) (Uses Equations (88) and (89)) are...

$$mean = \mathbb{E}\Big[\bar{y}_{t+1}\Big] = x_0 - (1-\theta)\,x_t - \alpha\,y_t \tag{90}$$

$$variance = \mathbb{E}\Big[\bar{y}^2_{t+1}\Big] - \Big\{\mathbb{E}\Big[\bar{y}_{t+1}\Big]\Big\}^2 = \mathbb{E}\Big[\bar{x}^2_{t+1}\Big] - \Big\{\mathbb{E}\Big[\bar{x}_{t+1}\Big]\Big\}^2 = \theta^2\sigma^2x_t^2 \tag{91}$$

Appendix

A. If $0 \le 1-\alpha-\theta \le 1$ then the equation for the sum of $1-\alpha-\theta$ over the time interval [0, t] is...

$$\begin{aligned} \sum_{i=0}^{t}(1-\alpha-\theta)^i &= \sum_{i=0}^{\infty}(1-\alpha-\theta)^i - (1-\alpha-\theta)^{t+1}\sum_{i=0}^{\infty}(1-\alpha-\theta)^i \\ &= \Big\{1-(1-\alpha-\theta)^{t+1}\Big\}\sum_{i=0}^{\infty}(1-\alpha-\theta)^i \\ &= \Big\{1-(1-\alpha-\theta)^{t+1}\Big\}\Big\{\frac{1}{1-(1-\alpha-\theta)}\Big\} \\ &= \frac{1}{\alpha+\theta}(1-(1-\alpha-\theta)^{t+1}) \end{aligned} \tag{92}$$

B. If $0 \leq 1-\alpha-\theta \leq 1$ then the equation for the sum of $1-\alpha-\theta$ over the time interval $[1,\text{t}]$ noting the results of the equation in Appendix A above is...

$$\begin{aligned}
\sum_{i=1}^{t}(1-\alpha-\theta)^i &= \left\{\sum_{i=0}^{t}(1-\alpha-\theta)^i\right\} - 1 \\
&= \frac{1}{\alpha+\theta}(1-(1-\alpha-\theta)^{t+1}) - 1 \\
&= \frac{1}{\alpha+\theta}((1-\alpha-\theta)-(1-\alpha-\theta)^{t+1}) \qquad (93)
\end{aligned}$$

C. The expected value of the random growth rate $\bar{x}_{t+1}$ as defined by Equation (86) is...

$$\begin{aligned}
\mathbb{E}\Big[\bar{x}_{t+1}\Big] &= \mathbb{E}\Big[(1-\theta\,(1+\sigma\,Z))\,x_t+\alpha\,y_t\Big] \\
&= \mathbb{E}\Big[x_t-\theta\,x_t-\theta\,\sigma\,Z\,x_t+\alpha\,y_t\Big] \\
&= \mathbb{E}\Big[x_t\Big]-\mathbb{E}\Big[\theta\,x_t\Big]-\mathbb{E}\Big[\theta\,\sigma\,Z\,x_t\Big]+\mathbb{E}\Big[\alpha\,y_t\Big] \\
&= x_t-\theta\,x_t-\theta\,\sigma\,x_t\mathbb{E}\Big[Z\Big]+\alpha\,y_t \\
&= (1-\theta)\,x_t+\alpha\,y_t \qquad (94)
\end{aligned}$$

D. The expected value of the random growth rate $\bar{x}_{t+1}$ squared as defined by Equation (86) is...

$$\begin{aligned}
\mathbb{E}\Big[\bar{x}_{t+1}^2\Big] &= \mathbb{E}\Big[(x_t-\theta\,x_t-\theta\,\sigma\,Z\,x_t+\alpha\,y_t)^2\Big] \\
&= \mathbb{E}\Big[x_t^2-2\theta x_t^2-2\theta\sigma Z x_t^2+2\alpha x_t y_t+\theta^2 x_t^2+2\theta^2\sigma Z x_t^2-2\theta\alpha x_t y_t+\theta^2\sigma^2 Z^2 x_t^2-2\theta\sigma\alpha Z x_t y_t+\alpha^2 y_t^2\Big] \\
&= \mathbb{E}\Big[x_t^2\Big]-\mathbb{E}\Big[2\theta x_t^2\Big]-\mathbb{E}\Big[2\theta\sigma Z x_t^2\Big]+\mathbb{E}\Big[2\alpha x_t y_t\Big]+\mathbb{E}\Big[\theta^2 x_t^2\Big]+\mathbb{E}\Big[2\theta^2\sigma Z x_t^2\Big]-\mathbb{E}\Big[2\theta\alpha x_t y_t\Big]+ \\
&\quad \mathbb{E}\Big[\theta^2\sigma^2 Z^2 x_t^2\Big]-\mathbb{E}\Big[2\theta\sigma\alpha Z x_t y_t\Big]+\mathbb{E}\Big[\alpha^2 y_t^2\Big] \\
&= x_t^2-2\theta x_t^2-2\theta\sigma x_t^2\mathbb{E}\Big[Z\Big]+2\alpha x_t y_t+\theta^2 x_t^2+2\theta^2\sigma x_t^2\mathbb{E}\Big[Z\Big]-2\theta\alpha x_t y_t+\theta^2\sigma^2 x_t^2\mathbb{E}\Big[Z^2\Big]-2\theta\sigma\alpha x_t y_t\mathbb{E}\Big[Z\Big]+\alpha^2 y_t^2 \\
&= x_t^2-2\theta x_t^2+2\alpha x_t y_t+\theta^2 x_t^2-2\theta\alpha x_t y_t+\theta^2\sigma^2 x_t^2+\alpha^2 y_t^2 \qquad (95)
\end{aligned}$$

E. The expected value of the random growth rate $\bar{y}_{t+1}$ as defined by Equation (87) (Uses Appendix Equation (94)) is...

$$\begin{aligned}
\mathbb{E}\Big[\bar{y}_{t+1}\Big] &= \mathbb{E}\Big[x_0-x_{t+1}\Big] \\
&= \mathbb{E}\Big[x_0\Big]-\mathbb{E}\Big[x_{t+1}\Big] \\
&= x_0-\mathbb{E}\Big[x_{t+1}\Big] \qquad (96)
\end{aligned}$$

F. The expected value of the random growth rate $\bar{y}_{t+1}$ squared as defined by Equation (87) (Uses Appendix Equation (95)) is...

$$\begin{aligned}\mathbb{E}\Bigg[\bar{y}_{t+1}\Bigg] &= \mathbb{E}\Bigg[(x_0 - x_{t+1})^2\Bigg] \\ &= \mathbb{E}\Bigg[x_0^2 - 2x_0 x_{t+1} + x_{t+1}^2\Bigg] \\ &= \mathbb{E}\Bigg[x_0^2\Bigg] - \mathbb{E}\Bigg[2x_0 x_{t+1}\Bigg] + \mathbb{E}\Bigg[x_{t+1}^2\Bigg] \\ &= x_0^2 - 2x_0\mathbb{E}\Bigg[x_{t+1}\Bigg] + \mathbb{E}\Bigg[x_{t+1}^2\Bigg]\end{aligned} \tag{97}$$